Anomalous scaling due to correlations: Limit theorems and self-similar processes

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We derive theorems which outline explicit mechanisms by which anomalous scaling for the probability density function of the sum of many correlated random variables asymptotically prevails. The results characterize general anomalous scaling forms, justify their universal character, and specify universality domains in the spaces of joint probability density functions of the summand variables. These density functions are assumed to be invariant under arbitrary permutations of their arguments. Examples from the theory of critical phenomena are discussed. The novel notion of stability implied by the limit theorems also allows us to define sequences of random variables whose sum satisfies anomalous scaling for any finite number of summands. If regarded as developing in time, the stochastic processes described by these variables are non-Markovian generalizations of Gaussian processes with uncorrelated increments, and provide, e.g., explicit realizations of a recently proposed model of index evolution in finance.

I. INTRODUCTION

A major achievement of the theory of probability are the limit theorems [1, 2], which provide the basis to explain statistical regularities observed in large classes of natural, economical and social mass-scale phenomena. These theorems describe the mechanisms leading to universal forms of scaling for the probability density functions (PDF's) of sums of many independent random variables. The scaling can be normal, or anomalous, depending on whether the PDF's of the individual variables possess finite second moment, or not. However, independence is not guaranteed in general, and a large number of collective phenomena in Nature exhibit anomalous scaling [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] as a consequence of correlations. In such cases, if the PDF of the sum of the elementary variables and its argument are simultaneously rescaled by a power D of the number of summands, it asymptotically converges to a scaling function g which is not necessarily Gaussian nor Lévy, and the scaling exponent D is in general not equal to $1/2$. Thus, an open challenge remains that of establishing limit theorems able to justify the existence and the universality of the anomalous scaling forms occurring in the case of strongly correlated variables.

The renormalization group approach to critical phenomena in statistical physics [3] has led to developments in probability theory which point towards a solution of this problem. Indeed, the fixed-point condition for block-spin transformations can be regarded [10, 11, 12] as a substitute of the stability condition at the basis of the limit theorems for the independent case [1, 2]. For instance, in the context of hierarchical equilibrium spin models the fixed-points of these block-spin transformations are expected to attract whole domains of strongly correlated critical systems displaying asymptotically the same universal form of anomalous scaling [10, 11, 12]. However, unlike in the case of the limit theorems for independent variables, classes of admissible universal scaling forms and their universality domains are not easily identified.

Since the standard limit theorems hold in force of the multiplicative structure of the joint PDF's of independent variables, an attempt has been recently made by the present authors [16] to establish theorems on the basis of a generalization of the multiplication operation, leading to dependent joint probability densities. Yet, due to mathematical difficulties, the problem of constructing consistent joint PDF's for correlated variables whose sum asymptotically satisfies scaling was not addressed [16].

Correlated random variables often considered in probability theory are those in exchangeable sequences [17]. The joint PDF's of an arbitrary number of variables in an exchangeable sequence have the property of being invariant under permutations of their arguments. Exchangeability was introduced by de Finetti [18], and is of paramount importance in the Bayesian approach to probability and statistics [17]. It is already known that, thanks to the simplifying feature of exchangeability, central limit theorems can be established [19]. The scalings foreseen by these theorems for the PDF of the sum of the random variables involve scaling functions which are convex combinations (mixtures) of Gaussians.

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For D only two values could be considered. If the variables are linearly uncorrelated, i.e., correlations are nonzero only for nonlinear functions of the variables, the scaling exponent is the ordinary $D = 1/2$ [19]. Alternatively, if the variables are correlated also at linear level, limit theorems have been proved for $D = 1$ [20].

Inspired by ideas from the modern theory of critical phenomena, in the present Article we establish limit theorems for sums of N dependent random variables whose joint PDF's, upon increasing N, do not define sequences of random variables, in general. With those defining exchangeable variable sequences, our joint PDF's only share the property of being invariant under permutations of their arguments. To illustrate how PDF's with such properties can arise in physics, we discuss the example of a permutationally invariant description of a magnetic system. The novel theorems apply to anomalous scalings with general exponent D. They also enable the explicit construction of universality domains, i.e. of whole classes of sequences of joint PDF's sharing asymptotically the same scaling form for the sum of the variables.

The limit theorems proved here have implications also outside the context of variables with permutationally invariant joint PDF's. Indeed, they were inspired by a recent proposal for the description of the time evolutions of financial indexes as stochastic processes [21, 22]. When dealing with such processes, one often considers time series in which each term represents the increment of an additive collective variable in an elementary time interval. Examples are the displacement in diffusion, or the logarithmic return of a financial asset. In these cases, causality imposes that the successive increments must constitute a sequence of random variables, in which the statistical properties of each variable are independent of the successive ones. When the increments are correlated and the processes have the property of self-similarity, i.e. when the collective variable distribution obeys scaling not just asymptotically, but for any finite number of summands, there are some requirements whose satisfaction has to be imposed to the joint PDF's of the successive increments. An heuristic way of satisfying these requirements was recently proposed as a basis for a stochastic model of the dynamics of financial indexes [21, 22]. As we show in this work, the heuristic proposal in [21, 22] is fully justified on the basis of the novel notion of stability implied by our theorems.

In general our stochastic processes are non-stationary and the scaling has a time-inhomogeneous nature [23]. When they become stationary, their increments also constitute sequences of exchangeable random variables. In such cases it is not possible to reproduce the statistics of these variables by empirical time-averages along infinitely long, single realizations of the processes. This is due to a mechanism of *ergodicity breaking* implied by de Finetti's representation theorem [17, 18]. A way out of this difficulty is found when considering self-similarity as a property of the process valid within a limited, although possibly large, range of time-scales. This attitude is fully legitimate in many applications [24]. We show here, by a dynamical simulation strategy of wide use in finance [25], how ergodicity can be restored in the process, by requiring scale-invariance to hold only up to a finite upper cutoff in time.

This Article is organized as follows. In the next three Sections, we introduce the formalism and present our main results about the limit theorems. We enunciate these theorems and give full details of their derivations in the Appendix. After stressing the applicability of our approach to the forms of anomalous scaling emerging, e.g., in the context of critical phenomena, in Sections V and VI we discuss implications of our results for the theory of stochastic processes. In particular, we present a class of non-Markovian self-similar processes possessing the requisites recently postulated [21, 22] for the case of finance and allowing explicit analytical calculations and efficient simulation strategies. The last Section is devoted to conclusions.

II. ANOMALOUS SCALING

Let us consider, for any given $N = 1, 2, 3, \ldots$, a set of random variables, X_i, with $i = 1, 2, \ldots, N$, taking values x_i on the real axis. We call $p_N(x_1, \ldots, x_N)$ the joint PDF of N-th set of variables and, to start with, assume that for any N this function is invariant under arbitrary permutations of its arguments. It should be stressed that, e.g., the random variable X_1 belonging to a set with N variables and the X_1 belonging to another set with $N' \neq N$ variables are not identical, in general. Thus, in principle we should denote the variables in the N-th set by $X_i^{(N)}$, $i = 1, 2, \ldots, N$, and their values by $x_i^{(N)}$. However, in order to keep formulas simple, we will not adopt this notation. Ultimately the identity of each variable X_i will be specified by the joint PDF $p_N(x_1, x_2, \ldots, x_i, \ldots, x_N)$ used in order to evaluate its statistical properties. In this way, our formulas will conform to the standards of the statistical mechanics literature [10, 11, 15]. To further simplify the formalism we can require, without loss of generality, that for any N all the variables have zero average, $\langle X_i \rangle_{p_N} = 0 \; \forall i$, where $\langle (\cdot) \rangle_{p_N} \equiv \int dx_1 \cdots dx_N \; (\cdot) \; p_N(x_1, \ldots, x_N)$. For the sum $Y_N \equiv X_1 + \cdots + X_N$, whose PDF is

$$p_{Y_N}(y) = \int dx_1 \cdots dx_N \; \delta(y - x_1 - \ldots - x_N) \; p_N(x_1, \ldots, x_N), \tag{1}$$

this also implies $\langle Y_N \rangle_{p_{Y_N}} = 0$. We are interested in cases in which the sequence $p_N(x_1, \ldots, x_N)$, $N = 1, 2, \ldots$ is such that p_{Y_N} satisfies anomalous scaling for $N \to \infty$, i.e.

$$N^D \, p_{Y_N}(N^D y) \to g(y), \tag{2}$$

where g is a scaling function, and D is a scaling dimension. We want to identify whole domains of p_N's such that the p_{Y_N} satisfies Eq. (2) with a given g and a given D. Besides the kind of convergence, the class of admissible g's and the range of D's needs to be specified. As we discuss below, examples of p_{Y_N}'s such that Eq.(2) holds are easily found in statistical physics.

We first clarify why the exponent values $D = 1/2$ and $D = 1$ naturally arise for sequences of exchangeable variables. Let us suppose that $\langle Y_N^2 \rangle_{p_N}$ is finite for any N. Since permutational invariance implies $\langle X_i \rangle_{p_N} = \langle X_1 \rangle_{p_N}$ $\forall i$, and $\langle X_i X_j \rangle_{p_N} = \langle X_1 X_2 \rangle_{p_N}$ $\forall i \neq j$, one has

$$\langle Y_N^2 \rangle_{p_N} = N \langle X_1^2 \rangle_{p_N} + N(N-1) \langle X_1 X_2 \rangle_{p_N}. \tag{3}$$

On the other hand, if, as appropriate for sequences of random variables, the sequence of joint PDF's p_N is constructed consistently with the condition

$$p_{N-1}(x_1, \ldots, x_{N-1}) = \int dx_N p_N(x_1, \ldots, x_N), \tag{4}$$

where $N \geq 2$, it is clear that $\langle X_1 \rangle_{p_N}$ and $\langle X_1 X_2 \rangle_{p_N}$ do not depend on N. Since according to the scaling condition in Eq. (2) $\langle Y_N^2 \rangle_{p_N} \sim N^{2D}$, Eq. (3) implies that either $D = 1/2$ and $\langle X_1 X_2 \rangle_{p_N} = 0$, or $D = 1$ and $\langle X_1 X_2 \rangle_{p_N} > 0$. In the former case, further restrictions on the averages of products of X's apply if higher moments of Y_N are assumed to exist. We should stress that if Eq.(4) is satisfied by the sequence of permutation-invariant joint PDF's, then these PDF's in turn define a sequence of exchangeable variables. Indeed, Eq.(4) guarantees that a given variable, say X_1, is strictly the same random variable, independent of the set of N variables within which it is considered.

As discussed in Section IV, there are cases, for example in statistical mechanics, where one considers a system in equilibrium at a given temperature, so that p_N represents the canonical joint PDF of N variables describing the degrees of freedom of the system. Since p_N is expressed as a ratio between the Gibbsian weight and the partition sum, upon integrating p_N over one of the N variables, as a rule we do not obtain the joint PDF of a system in equilibrium at the same temperature and with just $N - 1$ variables. Indeed, tracing over one of the variables leads to effective interactions which are not present in the Hamiltonian for $N - 1$ variables. The modern theory of critical phenomena shows that the renormalization effects determining this difference lead to anomalous scaling at the critical point [3]. This circumstance, which is expected to occur in many cooperative phenomena, will allow us to derive limit theorems for sums of exchangeable variables with general values of D.

On the other hand, in problems where N represents the number of increments over successive time intervals of a stochastic process and p_{N-1} and p_N are respectively the joint PDF's of the first $N - 1$ and N increments, causality imposes to consider sequences of p_N's satisfying Eq. (4). Below we will also show how the stability conditions implied by our limit theorems allow to define sequences of random variables whose joint PDF's satisfy Eq.(4) and whose aggregated increment Y_N satisfies anomalous scaling exactly for any N.

III. ILLUSTRATION OF THE MAIN RESULTS

We report our main statements and their mathematical proofs in the Appendix. Here we rather choose to illustrate the meaning and some implications of our results. Let us first consider p_N of the form

$$p_N(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{+\infty} d\mu \, \lambda(\mu) \, \prod_{i=1}^{N} \, l \left(x_i - \frac{\mu}{N^{1-D}} \right), \tag{5}$$

where λ and l are single-variable PDF's. With no loss of generality we require $\langle \mu \rangle_\lambda = 0$, whereas we assume $\langle X \rangle_l = 0$ and $\langle X^2 \rangle_l = 1$. The higher integer moments of l, are left arbitrary. Clearly, the p_N in Eq. (5) is a positive density normalized to 1, and invariant under permutations of its arguments. The X_i's are dependent, since p_N does not simply factorize into a product of single variable PDF's. The choice of considering p_N's which are convex combinations of products of single-variable PDF's is motivated by the fact that in this way it is possible to demonstrate the existence of asymptotic scalings with very general scaling functions. Indeed, in the Appendix we show that with the joint PDF

in Eq. (5), p_{Y_N} satisfies Eq. (2) with a scaling exponent $D \geq 1/2$. The scaling function g is determined by λ. For $D = 1/2$ g is given by

$$g(x) = \int_{-\infty}^{+\infty} d\mu \; \lambda(\mu) \; \frac{\exp\left[-(x-\mu)^2/2\right]}{\sqrt{2\pi}}, \tag{6}$$

whereas g coincides with λ itself if $D > 1/2$. In both cases, upon varying λ the scaling function g assumes general shapes. For instance, it may have several local and global maxima and power law decays to zero at large positive x, and/or $-x$, as required in many applications.

As anticipated above, when the variables X_i's are dependent and do not constitute a sequence, it is legitimate to introduce in the definition of p_N the N-dependence arising from the fact that μ enters divided by N^{1-D}. In particular, precisely this dependence implies that the joint PDF of a system with $N-1$ variables, p_{N-1}, rather than satisfying Eq. (4), is linked to p_N by the relation:

$$p_{N-1}(x_1,\ldots,x_{N-1}) = \left(\frac{N-1}{N}\right)^{(N-1)(1-D)} \int dx_N \, p_N\left(\left(\frac{N-1}{N}\right)^{(1-D)} x_1, \ldots, \left(\frac{N-1}{N}\right)^{(1-D)} x_{N-1}, x_N\right). \tag{7}$$

Consistently with the fact that the X_i's are not constituting a sequence of random variables, the marginal PDF of each individual X_i,

$$p_{X_i,N}(x_i) \equiv \int dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_N \, p_N(x_1,\ldots,x_N), \tag{8}$$

depends clearly on N ($N \geq i$). So, if the second moment of λ is finite, one realizes that $p_{X_i,N}$ has a finite width for $N \to \infty$ when $1/2 \leq D \leq 1$. If $D > 1$, this width diverges in the large N limit. Such a divergence makes full sense in a correlated context. Indeed, in relation to the anomalous character of the scaling, the marginal single-variable PDF's play here a role analogous to that of single-variable PDF's in the independent case. For example, with independent variables one allows the single variable PDF's to be of infinite width for any N, in order to have an anomalous, Lévy scaling limit [1, 2] of p_{Y_N}. Here, with correlated variables, the dependence on N entering in $p_{X_i,N}$ and the consequent divergence of width for $N \to \infty$ and $D > 1$ play a qualitatively similar role in producing anomalous scaling.

It is natural to ask what are the correlations of the variables X_i's according to the joint PDF's defined in Eq. (5). If $\langle \mu^2 \rangle_\lambda$ exists, an easy calculation gives for example

$$\langle X_i X_j \rangle_{p_N} = \frac{\langle \mu^2 \rangle_\lambda}{N^{2-2D}} \tag{9}$$

for $i \neq j$. In particular, the variables with permutation-invariant joint PDF's as in Eq. (5) are linearly correlated for finite N. When $1/2 \leq D < 1$ their linear correlators approach zero only asymptotically.

Next, we consider more general scaling functions which can be expressed as convex combinations of Gaussians with varying centers μ and widths σ. The form is

$$g(x) = \int_0^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\mu \, \psi(\sigma,\mu) \, \frac{\exp\left[-(x-\mu)^2/2\sigma^2\right]}{\sqrt{2\pi\sigma^2}}, \tag{10}$$

where $\sigma \in (0,\infty)$, and ψ is a PDF. The scaling exponent can be now any $D > 0$. Again, for the sake of simplicity we require $\langle \mu \rangle_\psi = 0$, while ψ must be strictly equal to zero in a whole neighborhood of $\sigma = 0$, for any μ. In the Appendix we prove that with the p_N's constructed as follows:

$$p_N(x_1,\ldots,x_N) = \int_0^{\infty} d\sigma \int_{-\infty}^{+\infty} d\mu \; \psi(\sigma,\mu) \prod_{i=1}^{N} \frac{l(x_i/\sigma N^{D-1/2} - \mu/\sigma N^{1/2})}{\sigma N^{D-1/2}}, \tag{11}$$

with $\langle X \rangle_l = 0$ and $\langle X^2 \rangle_l = 1$ as before, p_{Y_N} satisfies the asymptotic scaling (2) with g given by Eq. (10) and the chosen $D > 0$. One easily verifies that if $\psi(\sigma,\mu) = \rho(\sigma)\,\delta(\mu)$ the X variables are linearly uncorrelated for any N. In this case, with $D = 1/2$ the scaling limit of our theorem recovers known results valid for sums of random variables in exchangeable sequences [19].

It should also be noticed that if we put $\Psi(\sigma,\mu) = \delta(\sigma-1)\lambda(\mu)$ and $D = 1/2$, one recovers the case discussed at the beginning of this section.

IV. PERMUTATION-INVARIANT JOINT PDF'S AND CRITICAL PHENOMENA

All the cases discussed in the previous sections concern correlated variables whose joint PDF's for any N are permutationally invariant. At first sight, such feature may appear a too restrictive condition to be satisfied by realistic models, and applications may often require to release it. However, in the study of anomalous scaling variables of this kind may still play an important role. To illustrate this point, we consider the example of an Ising-like spin model, of the type often studied in the renormalization group approach to critical phenomena [3]. Let us consider a system of N spins S_i, $i = 1 \ldots, N$, where the index i labels the sites of a finite box of square or cubic lattice. The spins are supposed to take values s_i on the real axis. Equilibrium statistical mechanics allows in principle to construct the joint PDF of the N spin variables once given the spin Hamiltonian $H(\{s\})$ and the temperature T. Since the spin variables are associated to the lattice sites, their joint PDF is not invariant under permutations. Indeed, for any configuration $\{s_1, s_2, \ldots, s_N\}$, one has in general $H(s_{\pi(1)}, \ldots, s_{\pi(N)}) \neq H(s_1, \ldots, s_N)$, if π is a permutation of the N labels. This inequality holds because H is a sum of local interactions. Thus, also the canonical joint PDF

$$p'_N(s_1, \ldots, s_N) \equiv \frac{\exp[-H(s_1, \ldots, s_N)/k_B\,T]}{\int \prod_{i=1}^N ds'_i \exp[-H(\{s'\})/k_B\,T]} \tag{12}$$

where k_B is the Boltzmann constant, is not invariant under permutations of its arguments. On the other hand, when, e.g., discussing the critical behavior of the model, a key collective random quantity to be considered is the sum of all the spins $\sum_{i=1}^N S_i$ [3, 10, 11, 12], which, in contrast, is invariant under any permutation of the spin labels, and is expected to have a PDF satisfying anomalous scaling in the thermodynamic limit [10, 11, 15, 26]. This suggests to define what we call here a "permutation invariant representation" of the statistics of the model. Consider, for instance, the following definition of the joint PDF of new exchangeable variables X_i's:

$$p_N(x_1, x_2, \ldots, x_N) \equiv \frac{1}{N!} \sum_\pi \int \prod_{i=1}^N ds_i\, p'_N(s_1, s_2, \ldots, s_N) \prod_{j=1}^N \delta(x_j - s_{\pi(j)}), \tag{13}$$

where the sum is extended to all the $N!$ permutations π of the set $\{1, 2, \ldots, N\}$. The p_N's defined by the projection operation in Eq. (13) are indeed invariant under permutations, while their sum $Y_N = \sum_i X_i$ has a PDF identical to that of the total magnetization $\sum_i S_i$ of the original system. On the basis of the same projection, one can also define an effective Boltzmann factor for the variables X_i's in such a way that the partition function, and thus the free energy of the original problem, are preserved, too. Even if the computation of the effective Hamiltonian in terms of the X_i's is non-trivial, the above equations show that the asymptotic scaling of the PDF of $\sum_i S_i$ for a critical Ising-like model and that of $\sum_i X_i$ for its permutation invariant representation, coincide. It is also easy to see that one may construct different such representations of a given statistical model, all sharing the same free energy and the same PDF for Y_N.

For a critical Ising system one expects an anomalous scaling for the PDF of $\sum_{i=1}^N S_i$ with scaling dimensions $D = 15/16$ and $D \simeq 0.825$ for square and cubic lattices, respectively [3]. Taking into account that finite size scaling for the critical Ising model implies $\langle (\sum_i s_i)^2 \rangle_{p'_N} \sim N^{2D}$, one also concludes that for the permutation invariant representation defined by Eq. (13) one must have $\langle X_i X_j \rangle_{p_N} \sim N^{2D-2}$ for $N \to \infty$ and $i \neq j$. As a matter of fact, the limit theorem in Eq. (5) implies a scaling function for the PDF of Y_N and linear correlations for the X_i's (Eq. (9)) which are compatible with the asymptotic forms expected for the permutation invariant representation of the Ising model constructed here.

The above discussion clarifies that correlated variables with permutation-invariant PDF's can be relevant in the statistical approach to anomalous scaling. This relevance stems from the fact that for these variables the constructive limit theorems presented here are valid. At the same time, additive collective variables like the total magnetization of a critical Ising model are considered in many studies of complex systems, also outside equilibrium statistical mechanics [14].

V. NON-MARKOVIAN, SELF-SIMILAR STOCHASTIC PROCESSES

In many phenomena, anomalous scaling is a statistical symmetry obeyed to a good approximation for more or less broad ranges of finite N's. The validity of limit theorems of the kind proved in the previous sections opens the possibility of defining joint PDF's consistent with an exact anomalous scaling of p_{Y_N} for any finite N, i.e. such that $p_{Y_N} = N^D g(N^D y)$.

To illustrate how self-similarity for arbitrary finite N arises, let us consider the case of the central limit theorem for sums of independent random variables whose PDF has finite second moment. The asymptotic scaling is normal

and turns out to be an attractor in virtue of the stability property of the Gaussian PDF. In particular, this stability implies that if we consider a finite number of independent increments, X_1, X_2, ...,X_N, each one weighted by the same Gaussian PDF, the total increment $X_1 + X_2 + \cdots + X_N$ has also precisely a Gaussian PDF, having a width $N^{1/2}$ times the width of the individual increments. Thus, this PDF strictly satisfies normal scaling for any N.

In an analogous way, the results obtained in the previous sections for sums of correlated variables allow us to construct joint PDF's of the X variables consistent with an exact anomalous scaling of p_{Y_N}, for any finite N. The generalized stability conditions implied by our limit theorems make this possible. To be concrete, let us consider the case of the scaling function in Eq. (10). The construction of Eq. (11) implies that if we define

$$p_N(x_1, x_2, \ldots, x_N) = \int_0^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\mu \; \psi(\sigma,\mu) \prod_{i=1}^{N} \frac{\exp\left[-\left(x_i/\sigma N^{D-1/2} - \mu/\sigma N^{1/2}\right)^2/2\right]}{\sqrt{2\pi\sigma^2 N^{2D-1}}}, \tag{14}$$

this joint PDF is consistent with an exact anomalous scaling of p_{Y_N} with scaling function g given by Eq. (10) and exponent $D > 0$, for any finite N. Since at empirical level p_{Y_N} is often the most accessible PDF of the system [21, 22], such joint PDF's constructed in terms of g may be regarded as a model for the dependences determining the anomalous scaling in the range of N-values relevant for the phenomenon under study.

In the following, let us deal with processes developing in (discrete) time and think of X_i as an increment relative to the time interval $[(i-1)\Delta t, i\Delta t]$, while the elapsed time of the process is $t = N\Delta t$ and Δt is the elementary time-step of the process. Clearly, if p_N is the joint PDF of the first N increments of the same process developing in time, causality imposes the validity of Eq. (4) for any $N > 1$. The conditional PDF

$$p_N^c(x_N|x_1, x_2, \ldots, x_{N-1}) \equiv \frac{p_N(x_1, x_2, \ldots, x_N)}{p_{N-1}(x_1, x_2, \ldots, x_{N-1})} \tag{15}$$

($N \geq 2$), expresses the PDF of the N-th increment of the process, conditioned to the history of the previous $N-1$ ones. Like the joint PDF's, the conditional PDF's together with p_1 embody the full information on the process. For a causal process with non-Markovian character, a property we should be ready to give up for the X_i's is the invariance under permutations of their joint PDF's.

Referring again to an anomalous scaling with g as in Eq. (10) and $D > 0$, it is not difficult to figure out how to modify Eq. (14) in order to obtain a discrete-time stochastic process possessing self-similarity for finite N. To this purpose, let us introduce the following coefficients: $a_i \equiv [i^{2D} - (i-1)^{2D}]^{1/2}$ and $b_i \equiv i^D - (i-1)^D$, with $i = 1, 2, \ldots N, \ldots$. If we then define

$$p_N(x_1, x_2, \ldots, x_N) = \int_0^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\mu \; \psi(\sigma,\mu) \prod_{i=1}^{N} \frac{\exp\left[-(x_i - \mu b_i)^2/2\sigma^2 a_i^2\right]}{\sqrt{2\pi\sigma^2 a_i^2}}, \tag{16}$$

one can verify that this joint PDF indeed guarantees for any N a strict scaling for p_{Y_N}:

$$N^D p_{Y_N}(N^D y) = g(y). \tag{17}$$

Eq. (17) holds because the coefficients a_i and b_i satisfy $\sum_{j=1}^N a_j^2 = N^{2D}$ and $\sum_{j=1}^N b_j = N^D$, respectively. The condition in Eq. (4) is also respected. One recognizes immediately that for general $\psi(\sigma,\mu)$ the p_N's in Eq. (16) are not permutation invariant anymore for any $D > 0$. The lack of such invariance is also evident in the fact that $p_{X_i} \equiv p_{X_i,N}$ $\forall N \geq 1$ now varies with i, reflecting a nonstationarity of the increments.

When

$$\psi(\sigma,\mu) = \rho(\sigma)\,\delta(\mu) \tag{18}$$

with $\rho(\sigma) \neq \delta(\sigma_0)$, Y_N amounts to a stochastic processes of the form postulated recently for the description of financial indexes' evolution [21, 22]. In such a case, the increments are linearly uncorrelated and, up to an i dependent rescaling, their marginal PDF's coincide with g. The characteristic function of the scaling function g can be expressed as $\tilde{g}(k) = \int_0^\infty d\sigma \rho(\sigma) \exp(-\sigma k^2/2)$, and has the remarkable property that it is converted into to a proper N-dimensional joint characteristic function if k is replaced by $\sqrt{k_1^2 + k_2^2 + \cdots + k_N^2}$, for any N. Precisely this requirement has been identified in Refs. [21, 22] as a natural one for the joint characteristic function of the successive returns of an index. A theorem due to Schoenberg [17, 27] states that the $\tilde{g}(k)$'s having the above form exhaust the class of characteristic functions with such property. In particular, the class of scaling functions from which one can construct explicit joint PDF's is specified. This class includes the form used in Ref. [21] and also the Student distribution recently considered [32] in [28].

VI. RESTORING ERGODICITY

The ergodic properties of the dynamics of stochastic processes like those obtained using Eqs. (16,18) need to be analyzed in some detail. To be concrete, let us take ψ as in Eq. (18), with an arbitrary ρ and $D = 1/2$. In this particular case, since the a_i's are all equal, the increments constitute an exchangeable sequence and are stationary. Hence, the problem of ergodicity is clearly posed. The form of the joint PDF's in Eq. (16) amounts to a convex combination of uncorrelated Gaussian increments with different σ's. Any simulation of a single, infinitely long history $(x_1, x_2, \ldots, x_N, \ldots)$ made on the basis of the sequence $p_1(x_1) = g(x_1)$, $p_2^c(x_2|x_1)$, $\ldots$, $p_N^c(x_N|x_{N-1}, \ldots, x_1)$, $\ldots$ would not be apt to manifest the ensemble correlations implied by p_N in Eq. (16). Indeed, after an initial transient, the extraction of the successive increments would essentially be ruled by a Gaussian conditional PDF with an approximately constant $\sigma = \overline{\sigma}$, chosen among all those allowed by ρ. A different simulation would pick up a different $\overline{\sigma}$ in the initial transient stage and then proceed with independent increments extracted according to this $\overline{\sigma}$ (see Appendix). The correlations implied by Eq. (16) are reproduced only by putting together the results of an ensemble of a large number of different such simulations. A sliding time-interval sampling procedure along a single infinite history would not detect any correlations among the increments. This amounts to a breaking of ergodicity: The single infinitely-long realization of the process just isolates one of its possible uncorrelated ergodic components, a well known consequence of de Finetti's representation theorem for exchangeable variable sequences [17]. This lack of ergodicity appears at first sight to represent a serious limitation of the stochastic process, if like in finance a legitimate ambition is to simulate single long histories with the same correlation and scaling properties as the empirical one.

It is possible to recover the anomalous scaling and the correlations implied by our construction of the joint PDF's using a suitably defined dynamics. Let us go back to the motivations mentioned above for considering self-similar processes: The approximate satisfaction of anomalous scaling for PDF's like that of the aggregated increment in a time interval of duration τ is often valid for a limited range, $\tau \leq M\,\Delta t$. Under these premises, an adequate goal for the simulation is that of reproducing, by time-averages along a single dynamical trajectory, the scaling and correlation properties implied by Eq. (16) just over the time range $M\,\Delta t$. One way of obtaining these properties, namely ergodicity and self-similarity up to the time-scale $M\,\Delta t$, is by implementing an autoregressive dynamics [25] with memory span equal to M. Imagine we have extracted, consistently with the conditional PDF's p_i^c, $i = 1, 2, \ldots, M$, the first M increments of the additive variable Y_M. Instead of using the conditional PDF $p_{M+1}^c(x_{M+1}|x_M, x_{M-1}, \ldots, x_1)$ to extract the $M+1$-th increment, we use $p_M^c(x_{M+1}|x_M, x_{M-1}, \ldots, x_2)$. Similarly, for any time $t > M\,\Delta t$ we use this autoregressive scheme in which only the preceding $M-1$ increments have an effect on the further evolution. In this way one circumvents the problem of broken ergodicity, because for finite M the conditioning input is constantly updated and modified to an extent which is sufficient for a long-enough simulation to span all the σ's allowed by the ensemble in Eq. (16). With such strategy the empirical PDF of the sum of the increments over an interval τ, sampled from all intervals of duration τ along a single long history of the process, satisfies to a very good approximation the anomalous scaling for $\tau \leq M\,\Delta t$ (see Appendix).

VII. CONCLUDING REMARKS AND PERSPECTIVES

In this Article we have shown that the choice of variables with joint PDF's invariant under permutations is particularly favorable for discussing the problem of the asymptotic emergence and universality of anomalous scaling due to correlations. Ideas of the modern theory of critical phenomena and complex systems are at the basis of the advancements we could present here. Our limit theorems cover indeed forms of anomalous scaling, which, to our best knowledge, so far have not been treated by the probabilistic literature with the present generality. At the same time, classical examples taken from the theory of critical phenomena gave us a way to illustrate the role variables with permutation invariant joint PDF's can play in more general problems with anomalous scaling.

As remarked above, the idea of basing limit theorems for correlated variables on some suitable generalization of the standard multiplication has some appeal [16]. The rules by which we compose the l PDF's to obtain p_N in Eqs. (5) or (11), retain in fact the commutative and associative properties. In this respect, our approach to anomalous scaling is quite different from the renormalization group one, and remains closer in spirit to the limit theorems for independent variables. This closeness is also manifest in the relative simplicity of our proofs, which directly rely on the corresponding ones for the independent case. Thus, the mathematics at the basis of the standard central limit theorem plays a fundamental role also outside the context of independent variables. Another difference of our approach compared to the renormalization group is that we do not need to make use of the hierarchical modeling to have analytical control on statistical coarse-graining operations. Here we replace the hierarchical paradigm by the assumption of invariance under permutations. In principle, this replacement still allows to address realistic scalings as illustrated in Section IV.

We have expressed our limit PDF's for the (rescaled) sums of correlated random variables as convex combinations

of Gaussians with varying widths and/or centers. Scaling functions belonging to this class have been considered very often in phenomenological descriptions of anomalous scaling [29], but their possible implications as far as correlations are concerned were not stressed enough, in our opinion. The wide classes of scaling functions and the continuous ranges of scaling exponents identified through our theorems, definitely do not support the idea that in the context of strongly correlated variables relevant scaling forms could be organized in a restricted set of universality classes. In particular, there does not appear to exist one or few particular scaling functions playing a universal role similar to the one of the Gaussian in the independent case.

The generalization of the notion of stability implied by our theorems naturally leads to the introduction of self-similar stochastic processes with correlated increments. These include in particular the process proposed in Ref. [21] as a model of index evolution in finance. Besides giving this proposal a rigorous basis, the results presented here, especially those concerning the restoration of ergodicity, substantially enhance the analytical and numerical tractability of such a process.

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VIII. APPENDIX

In the first part of this Appendix, we prove three different statements which in particular imply that p_{Y_N}, the PDF of $Y_N \equiv \sum_{i=1}^N X_i$ satisfies the scaling

$$N^D\, p_{Y_N}(N^D y) \to g(y), \tag{19}$$

for $N \to \infty$ (refer to main text for details).

Limit Theorem for g's given by Gaussian mixtures with different centers and $D = 1/2$
Given the sequence of joint PDF's

$$p_N(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{+\infty} d\mu\ \lambda(\mu) \prod_{i=1}^N l\left(x_i - \frac{\mu}{N^{1-D}}\right), \quad N = 1, 2, \ldots \tag{20}$$

for the random variables $\{X_i\}_{i=1,2,\ldots,N}$, where $D = 1/2$, λ and l are single-variable PDF's with $\langle \mu \rangle_\lambda = 0$ and $\langle X \rangle_l = 0$, $\langle X^2 \rangle_l = 1$, then as $N \to \infty$ the probability

$$Prob\left\{\sum_{i=1}^N \frac{X_i}{N^D} < z\right\} \to \int_{-\infty}^{z} dw\ g(w) \tag{21}$$

uniformly, with

$$g(w) = \int_{-\infty}^{+\infty} d\mu\ \lambda(\mu)\ \frac{\exp\left[-(w-\mu)^2/2\right]}{\sqrt{2\pi}}. \tag{22}$$

Let us consider the positive quantity

$$Prob\left\{\sum_{i=1}^N \frac{X_i}{N^{1/2}} < z\right\}_\mu \equiv \int_{-\infty}^{z} dw\ \int \prod_{i=1}^N dx_i\ l\left(x_i - \frac{\mu}{N^{1/2}}\right)\ \delta\left(w - \sum_{i=1}^N \frac{x_i}{N^{1/2}}\right), \tag{23}$$

which, once multiplied by λ and integrated with respect to μ, yields the probability that $\sum_i X_i/N^{1/2} \le z$. The following identity holds:

$$Prob\left\{\sum_{i=1}^N \frac{X_i}{N^{1/2}} < z\right\}_\mu = Prob\left\{\sum_{i=1}^N \frac{X_i}{N^{1/2}} < z - \mu\right\}_0 . \tag{24}$$

The central limit theorem for independent variables guarantees [1, 2] that the right hand side of Eq. (24) converges uniformly to

$$\int_{-\infty}^{z} dw \frac{\exp\left[-(w-\mu)^2/2\right]}{\sqrt{2\pi}}. \tag{25}$$

Since the uniform convergence holds for z and μ separately, we can interchange the integration in μ with the limit for $N \to \infty$ and get

$$\int_{-\infty}^{+\infty} d\mu \lambda(\mu) Prob\left\{\sum_{i=1}^{N} \frac{X_i}{N^{1/2}} < z\right\}_{\mu} \to \int_{-\infty}^{z} dw \int_{-\infty}^{+\infty} d\mu \lambda(\mu) \frac{\exp\left[-(w-\mu)^2/2\right]}{\sqrt{2\pi}}, \tag{26}$$

still uniformly in z. This proves the asymptotic scaling (19) of p_{Y_N}, with $D = 1/2$ and g as in Eq. (20).

Limit Theorem for g's given by Gaussian mixtures with different centers and $D > 1/2$
Here, we establish a similar result with $D > 1/2$. Let us look back at Eq. (20), where we have a convex combination of Gaussians with finite second moment equal to 1. Suppose to perform a limit in which this second moment is sent to zero: In this limit the Gaussian would approach a Dirac delta-function. Hence, we would have

$$g(x) = \lambda(x). \tag{27}$$

In order to construct p_N such that p_{Y_N} satisfies asymptotic scaling with $g = \lambda$ and with $D > 1/2$, it is convenient to consider the characteristic functions of p_N, and of g, respectively:

$$\begin{aligned} \tilde{p}_N(k_1, \ldots, k_N) &= \int \prod_{i=1}^{N} dx_i \exp(-ik_i x_i)\, p_N(x_1, \ldots, x_N), && (28) \\ \tilde{g}(k) &= \int dw \exp(-ikw)\, g(w). && (29) \end{aligned}$$

We can prove the following statement.
Given the sequence of joint PDF's

$$p_N(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{+\infty} d\mu\, \lambda(\mu) \prod_{i=1}^{N} l\left(x_i - \frac{\mu}{N^{1-D}}\right), \quad N = 1, 2, \ldots \tag{30}$$

for the random variables $\{X_i\}_{i=1,2,\ldots,N}$*, where* $D > 1/2$*,* λ *and* l *are single-variable PDF's with* $\langle\mu\rangle_\lambda = 0$ *and* $\langle X\rangle_l = 0$*,* $\langle X^2\rangle_l = 1$*, then as* $N \to \infty$ *we have*

$$\tilde{p}_N\left(\frac{k}{N^D}, \frac{k}{N^D}, \ldots, \frac{k}{N^D}\right) \to \tilde{g}(k), \tag{31}$$

with

$$g(w) = \lambda(w). \tag{32}$$

The convergence is uniform in k *if* $|\tilde{\lambda}(k)|$ *decays at large* $|k|$ *as* $1/|k|^2$ *or faster, uniform in* k *in any bounded subset of* $\mathbb{R}$ *otherwise.*

Indeed, the characteristic function of such p_N is

$$\tilde{p}_N(k_1, \ldots, k_N) = \int_{-\infty}^{+\infty} d\mu\, \lambda(\mu) \exp\left[-i(k_1 + \cdots + k_N)\frac{\mu}{N^{(1-D)}}\right] \prod_{i=1}^{N} \tilde{l}(k_i), \tag{33}$$

where $\tilde{l}$ is the characteristic function of l. We can write

$$\tilde{p}_N\left(\frac{k}{N^D}, \frac{k}{N^D}, \ldots, \frac{k}{N^D}\right) = \int_{-\infty}^{+\infty} d\mu\, \lambda(\mu) \exp(-ik\mu)\, \tilde{l}(k/N^D)^N. \tag{34}$$

If we assume $D > 1/2$, $\tilde{l}(k/N^D)^N$ approaches 1 for $N \to \infty$, uniformly in k in any bounded subset of $\mathbb{R}$. This implies that as $N \to \infty$,

$$\tilde{p}_N \left(\frac{k}{N^D}, \frac{k}{N^D}, \dots, \frac{k}{N^D} \right) \to \tilde{\lambda}(k), \tag{35}$$

which proves the theorem. The convergence in Eq. (35) is uniform in k if $|\tilde{\lambda}(k)|$ decays at large $|k|$ as $1/|k|^2$ or faster, uniform in k in any bounded subset of $\mathbb{R}$ otherwise.

Limit Theorem for g's given by Gaussian mixtures with different centers and widths, and $D > 0$
Given the sequence of joint PDF's

$$p_N(x_1, \dots, x_N) = \int_0^\infty d\sigma \int_{-\infty}^{+\infty} d\mu \, \psi(\sigma, \mu) \prod_{i=1}^N \frac{l(x_i/\sigma N^{D-1/2} - \mu/\sigma N^{1/2})}{\sigma N^{D-1/2}}, \tag{36}$$

for the random variables $\{X_i\}_{i=1,2,\dots,N}$, where $D > 0$, ψ is a joint PDF identically equal to zero in a whole neighborhood of $\sigma = 0$ and such that $\langle \mu \rangle_\psi = 0$, and l is a single-variable PDF with $\langle X \rangle_l = 0$, $\langle X^2 \rangle_l = 1$, then as $N \to \infty$ the probability

$$Prob \left\{ \sum_{i=1}^N \frac{X_i}{N^D} < z \right\} \to \int_{-\infty}^z dw \; g(w) \tag{37}$$

uniformly, with

$$g(w) = \int_0^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\mu \, \psi(\sigma, \mu) \, \frac{\exp\left[-(w-\mu)^2/2\sigma^2\right]}{\sqrt{2\pi\sigma^2}}. \tag{38}$$

For any σ and μ we define the quantity

$$\begin{aligned} Prob \left\{ \sum_{i=1}^N X_i/N^D < z \right\}_{\sigma,\mu} &\equiv \int_{-\infty}^z dw \int_{-\infty}^{+\infty} dx_1 \dots dx_N \delta \left(w - \frac{x_1 + \dots + x_N}{N^D} \right) \prod_{i=1}^N \frac{l(x_i/\sigma N^{D-1/2} - \mu/\sigma N^{1/2})}{\sigma N^{D-1/2}} \\ &\equiv f\left(z, N, D, \mu, \sigma\right) \end{aligned} \tag{39}$$

One easily verifies the following property:

$$f\left(z, N, D, \mu, \sigma\right) = f\left(\frac{z}{\sigma} - \frac{\mu}{\sigma}, N, 1/2, 0, 1 \right). \tag{40}$$

Under the present assumptions, the central limit theorem for independent variables [1, 2] guarantees that the quantity on the right hand side of Eq. (40) converges uniformly to the limit

$$\int_{-\infty}^z dw \frac{\exp(-(w-\mu)^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}. \tag{41}$$

In view of the conditions on ψ, this uniformity holds for σ, μ and z, separately. We thus conclude that

$$\int_0^\infty d\sigma \int_{-\infty}^{+\infty} d\mu \, \psi(\sigma, \mu) \, Prob \left\{ \sum_{i=1}^N X_i/N^{1/2} < z \right\}_{\sigma,\mu} \to \int_{-\infty}^z dw \int_0^\infty d\sigma \int_{-\infty}^{+\infty} d\mu \, \psi(\sigma, \mu) \, \frac{\exp(-(w-\mu)^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}, \tag{42}$$

uniformly in z. The uniformity follows from the hypothesis that ψ is zero in a whole neighborhood of $\sigma = 0$.

If $\psi(\sigma, \mu) = \rho(\sigma)\delta(\mu)$, i.e. with g given by a mixture of Gaussians of different widths and all centered in the origin, the X variables are linearly uncorrelated for any N.

Simulating a self-similar process with strongly correlated increments
Here we discuss the issue of how to simulate the stochastic processes described in the main text. Let X_i be the

increment relative to the time interval $[(i-1)\Delta t, i\Delta t]$ of the discrete-time process $Y_N \equiv \sum_{i=1}^{N} X_i$, where Δt is an elementary time-step and $t \equiv N\,\Delta t$ the elapsed time. For the sake of definiteness, we assume ψ to be of the form

$$\psi(\sigma,\mu) = \delta(\mu)\,\rho(\sigma), \tag{43}$$

with

$$\rho(\sigma) = A\frac{6b^3\sigma^2}{\pi\,(b^6+\sigma^6)} \tag{44}$$

for $\sigma \in (\sigma_{min}, \sigma_{max})$ $(0 < \sigma_{min} < \sigma_{max})$ and $\rho(\sigma) = 0$ elsewhere. The parameter A is just a normalization constant fixed such as $\int_{\sigma_{min}}^{\sigma_{max}} d\sigma\,\rho(\sigma) = 1$, whereas b determines $\langle\sigma^2\rangle_\rho$. With such a choice, the scaling function

$$g(x) = \int_{\sigma_{min}}^{\sigma_{max}} d\sigma\;\frac{6b^3\sigma^2}{\pi\,(b^6+\sigma^6)}\;\frac{\exp\left[-x^2/2\sigma^2\right]}{\sqrt{2\pi\sigma^2}}, \tag{45}$$

is even. With a sufficiently large σ_{max}, we can mimic a fat-tail power-law decay for g of the kind $g(x) \sim 1/|x|^4$ at large arguments. We first address the situation in which the problem of breaking of ergodicity is well posed, i.e., when the increments X_i's of the process are stationary, so that it makes sense to compare their ensemble and time averages. Later, we will comment about the more general case. We thus fix $D = 1/2$. With the choice (43) for ψ, this also implies that the X_i's are exchangeable. Indeed, according to Eq. (16) of the main text, the joint PDF for the increments of the process becomes

$$p_N(x_1,\ldots,x_N) = \int_{\sigma_{min}}^{\sigma_{max}} d\sigma\frac{6b^3\sigma^2}{\pi\,(b^6+\sigma^6)}\frac{e^{-(x_1^2+\cdots+x_N^2)/2\sigma^2}}{(2\pi\sigma^2)^{N/2}}, \tag{46}$$

and a straightforward calculation yields

$$\begin{aligned} C_{\alpha\beta}(i,j) &\equiv \frac{\langle|X_i|^\alpha|X_j|^\beta\rangle_{p_N} - \langle|X_i|^\alpha\rangle_{p_1}\langle|X_j|^\beta\rangle_{p_1}}{\langle|X_i|^{\alpha+\beta}\rangle_{p_1} - \langle|X_i|^\alpha\rangle_{p_1}\langle|X_i|^\beta\rangle_{p_1}} && (47)\\ &= \frac{B_\alpha B_\beta\;\left[\langle\sigma^{\alpha+\beta}\rangle_\rho - \langle\sigma^\alpha\rangle_\rho\langle\sigma^\beta\rangle_\rho\right]}{B_{\alpha+\beta}\langle\sigma^{\alpha+\beta}\rangle_\rho - B_\alpha B_\beta\;\langle\sigma^\alpha\rangle_\rho\langle\sigma^\beta\rangle_\rho} && (48)\end{aligned}$$

$\forall\; i,j = 1,2,\ldots,N$, with

$$B_\alpha \equiv \int_{-\infty}^{+\infty} dx\;|x|^\alpha\;\frac{e^{-x^2/2}}{\sqrt{2\pi}}. \tag{49}$$

Notice that when $\sigma_{max} \to \infty$, $\langle\sigma^{\alpha+\beta}\rangle_\rho$ is finite only for $\alpha+\beta < 3$. The strong correlations among the increments are reflected by the fact that the $C_{\alpha\beta}(i,j)$ is different from zero. On the other hand, $\langle X_iX_j\rangle_{p_N} = 0\;\forall\; j \neq i$, and the process is uncorrelated at linear level. Hence,

$$C_{lin}(i,j) \equiv \frac{\langle X_iX_j\rangle_{p_N} - \langle X_i\rangle_{p_1}\langle X_j\rangle_{p_1}}{\langle X_i^2\rangle_{p_1} - \langle X_i\rangle_{p_1}\langle X_i\rangle_{p_1}} \tag{50}$$

is equal to 1 for $j = i$, and zero otherwise.

A natural strategy of simulation of the process is based on extracting the random increments $x_1, x_2, \ldots, x_N, \ldots$ according to the sequence of conditional PDF's

$$p_1(x_1) = g(x_1),\; p_2^c(x_2|x_1),\;\ldots,\; p_i^c(x_i|x_{i-1},\ldots,x_1),\;\ldots, \tag{51}$$

respectively. To distinguish with what follows, we call this simulation scheme "progressive". An ensemble of a large number of independent simulations of this kind reproduces well all the theoretical features of the process. For instance, in Fig. 1a we find that $C_{lin}(1,j)$ and $C_{1,1}(1,j)$ obtained from an ensemble of 10^5 simulations oscillate around the correct theoretical values. On the contrary, if we consider a single simulation of N steps $(N \gg 1)$ generated according to Eq. (51), and the associated "sliding-window" correlators

$$\begin{aligned} \overline{C}_{\alpha\beta}(k) &\equiv \frac{\frac{1}{N-k}\sum_{i=1}^{N-k}|x_i|^\alpha|x_{i+k}|^\beta - \left(\frac{1}{N}\sum_{i=1}^{N}|x_i|^\alpha\right)\left(\frac{1}{N-k}\sum_{i=1}^{N-k}|x_{i+k}|^\beta\right)}{\frac{1}{N}\sum_{i=1}^{N}|x_i|^{\alpha+\beta} - \left(\frac{1}{N}\sum_{i=1}^{N}|x_i|^\alpha\right)\left(\frac{1}{N}\sum_{i=1}^{N}|x_i|^\beta\right)}, && (52)\\ \overline{C}_{lin}(k) &\equiv \frac{\frac{1}{N-k}\sum_{i=1}^{N-k}x_ix_{i+k} - \left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)\left(\frac{1}{N-k}\sum_{i=1}^{N-k}x_{i+k}\right)}{\frac{1}{N}\sum_{i=1}^{N}x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)\left(\frac{1}{N}\sum_{i=1}^{N}x_i\right)}, && (53)\end{aligned}$$

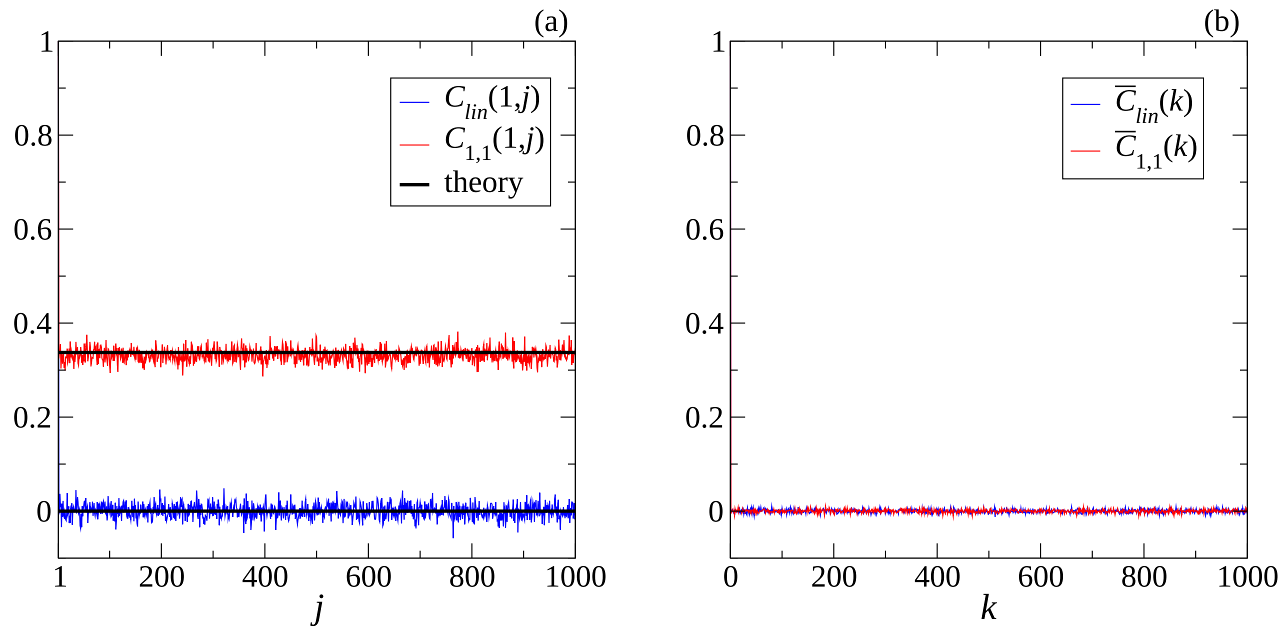

FIG. 1: Ergodicity breaking for a progressive simulation. Correlations calculated from an ensemble of 10^5 realizations (a) and from a single realization of 10^5 steps (b). Here, and in the following we use $\rho(\sigma)$ as in Eq. (44), with $\sigma_{min} = 0.01$, $\sigma_{max} = 10$, and $b = 1/\sqrt{2}$.

we find that both $\overline{C}_{1,1}(k)$ and $\overline{C}_{lin}(k)$ are zero for $k > 0$ (see Fig. 1b). This means that time-averages disagree with ensemble-averages, i.e., the dynamics is not ergodic.

We gain an insight into this ergodicity breaking by noticing that the conditional PDF for the next increment at each time-step i can be expressed in the following way:

$$p_i^c(x_i|x_{i-1},\ldots,x_1) = \int_{\sigma_{min}}^{\sigma_{max}} d\sigma\ \rho_i^c(\sigma|x_{i-1},\ldots,x_1)\ \frac{e^{-x_i^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \tag{54}$$

where the conditional PDF for the value σ, ρ_i^c, is in fact a function f_i depending only on $\sum_{j=1}^{i-1} x_j^2$:

$$\rho_i^c(\sigma|x_{i-1},\ldots,x_1) = \frac{\frac{\rho(\sigma)\ \prod_{j=1}^{i-1} e^{-x_j^2/2\sigma^2}}{\sigma^M}}{\int_0^{+\infty} d\sigma'\ \frac{\rho(\sigma')\ \prod_{j=1}^{i-1} e^{-x_j^2/2\sigma'^2}}{\sigma'^M}} \equiv f_i\left(\sigma, \sum_{j=1}^{i-1} x_j^2\right). \tag{55}$$

As i increases, very quickly f_i becomes sharply peaked around a specific value $\overline{\sigma}$, which depends on the sum of the squares of the past increments, $\sum_{j=1}^{i-1} x_j^2$. For a given $i \gg 1$, the dynamics is such that the typical growth of $\sum_{j=1}^{i} x_j^2$ with respect to $\sum_{j=1}^{i-1} x_j^2$ compensates for the functional change of f_{i+1} with respect to f_i, and the new conditional PDF, ρ_{i+1}^c, remains peaked around the same value $\overline{\sigma}$. In this way, a single ergodic component labeled by $\overline{\sigma}$ is chosen during the initial stages of the simulation, when ρ_i^c still resembles ρ. The subsequent dynamical evolution is then very similar to a process with independent increments at the initially selected $\overline{\sigma}$. This is why along a single history of the process the sliding-window analysis performed in Fig. 1b reveals a vanishing $\overline{C}_{1,1}(k)$ for $k > 0$.

In practice, a progressive simulation scheme can be realized by first extracting a σ according to the PDF in Eq. (55), and then an x_i from a Gaussian PDF with width σ. With a different, autoregressive, simulation strategy, scaling and ergodic properties can be restored together within a good approximation up to a finite time-scale M. This is obtained by considering a conditional PDF $p_i^{c,ar}$ which depends, still through Eq. (54), on the previous $M-1$ increments only, for all $i \geq M$:

$$p_i^{c,ar}(x_i|x_{i-1},\ldots,x_{i-M+1}) \equiv p_M^c(x_i|x_{i-1},\ldots,x_{i-M+1}). \tag{56}$$

After the initial transient of M time-steps, which is realized according to the progressive scheme in Eqs. (51), using $p_i^{c,ar}$ at each step $i \geq M$ we "forget" the increment x_{i-M} and we thus fix to M the dimension of the conditional

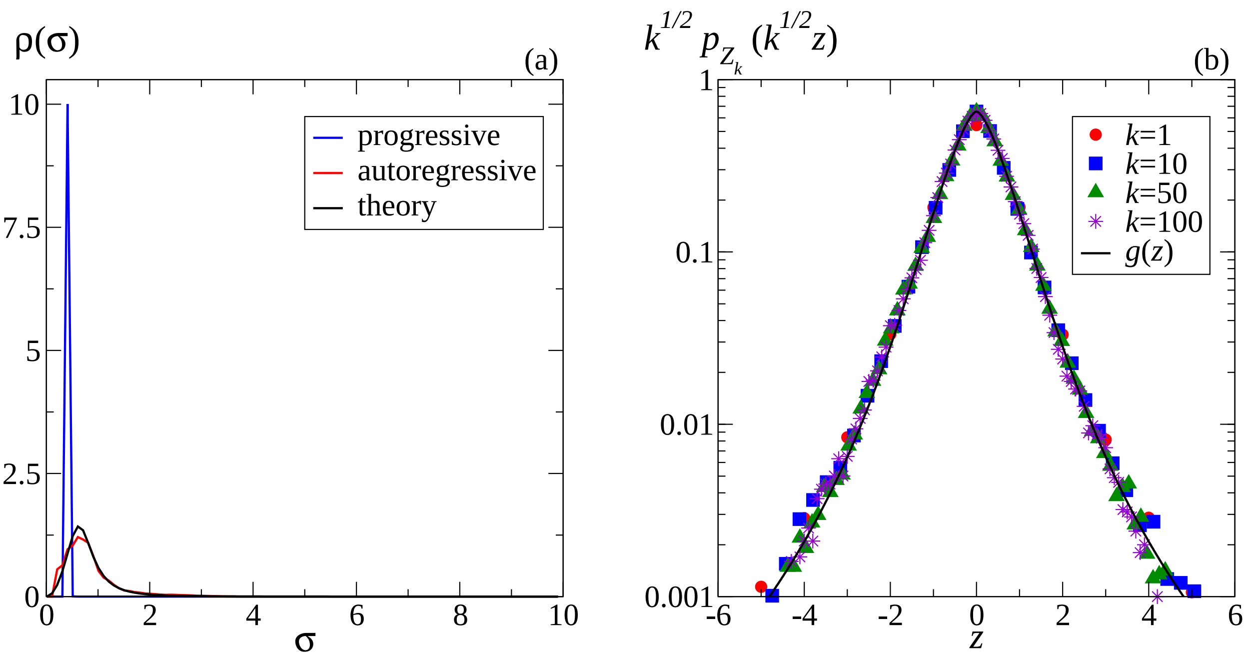

FIG. 2: (a) Histogram of the frequency of ergodic components σ in ρ_i^c: Only for the autoregressive simulation the histogram reproduces well $\rho(\sigma)$ in Eq. (44). (b) The rescaling of the histogram of the increments over an interval of duration $\tau = k\,\Delta t$ for a single autoregressive simulation of 10^5 time-steps with $M = 100$ reproduces $g(z)$ for $k \leq M$.

PDF for extracting the next increment of the process. This enables the conditional PDF ρ_M^c to wander among all the ergodic components labeled by the different σ's, as it is shown in Fig. 2a, where we recorded the histogram of the σ's in Eqs. (54,56) spanned by both a progressive and an autoregressive ($M = 100$) simulation of 10^5 time-steps. While in the progressive case the histogram is strongly peaked around a single $\overline{\sigma}$, in the autoregressive one it well reproduces the $\rho(\sigma)$ assumed in Eq. (44). We define the increment over an interval of duration $\tau = k\,\Delta t$ at time $t = i\,\Delta t$ as $Z_{ik} \equiv Y_{i+k} - Y_i$ $(i = 1, 2, \ldots, N-k,\ k > 0)$, and then we sample the the PDF $p_{Z_k}(z) \equiv \frac{1}{N-k}\sum_{i=1}^{N-k} p_{Z_{ik}}(z)$ along a single autoregressive history of N steps. For an ergodic dynamics it is expected that the scaling properties of p_{Z_k} reproduce those of p_{Y_k}. Fig. 2b shows that indeed the desired scaling properties for p_{Z_k},

$$k^D\, p_{Z_k}(k^D z) = g(z), \tag{57}$$

are well satisfied for $D = 1/2$ and $k \leq M$. The fidelity and the ergodicity of the autoregressive simulation are furthermore supported by an inspection of $C_{\alpha\beta}(i,j)$ and $\overline{C}_{\alpha\beta}(k)$, which reveals that both the ensemble and the time correlations approximatively coincide with the theoretical values as long as $j-i$ and k are smaller than M, respectively (Fig. 3a,b). For larger time separations, correlations slowly decay to zero, producing a smooth crossover to a process with independent increments on scales much larger than M.

For simulations with $D \neq 1/2$, by considering the rescaled variables $X_1 i' \equiv X_i/a_i$, with $a_i \equiv \left[i^{2D} - (i-1)^{2D}\right]^{1/2}$ (see main text), the above discussion still applies. As a consequence, the mechanism of the selection of a specific value $\sigma = \overline{\sigma}$ in ρ_i^c for a single progressive simulation and that of the dynamical sampling of the various σ's for a single autoregressive one remain valid also when $D \neq 1/2$. These features are of crucial importance for the applicability of such kind of processes in finance [23].

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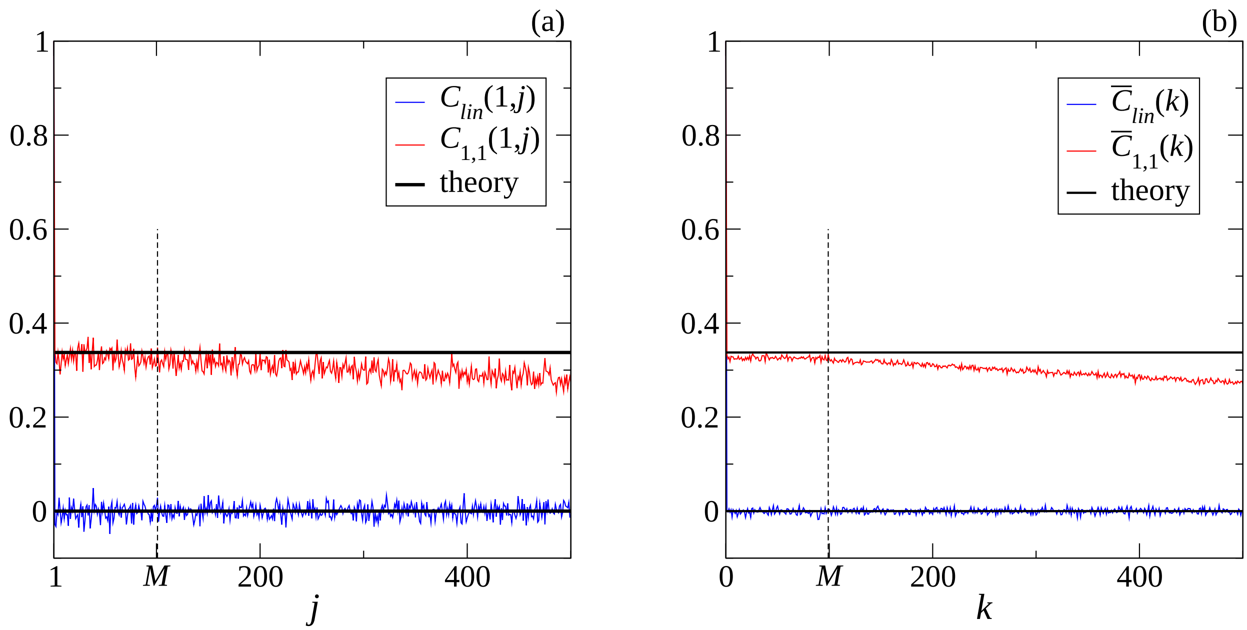

FIG. 3: An autoregressive simulation restores ergodicity up to the time-scale $\tau = M\,\Delta t$. Ensemble (a) and time (b) correlations are calculated as in Fig. 1.

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